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Pseudo-Differential Operators with Nonregular Symbols

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1. INTRODUCTION

The purpose of this paper is to investigate the boundedness of the pseudo-differential operators having symbols which do not have bounded derivatives with respect to the space variables. An operator P from $C_0^\infty(R^n)$ to $C^0(R^n)$ is called a pseudo-differential operator with symbol $p(x, \xi)$ if

$$Pu(x_1, x_2, ..., x_n) = \int e^{i(x_1\xi_1+\cdots+x_n\xi_n)} p(x_1, ..., x_n; \xi_1, ..., \xi_n)\, \hat{u}(\xi_1, ..., \xi_n)\, d\xi_1, ..., d\xi_n,$$

where a power of 2π is ignored and $\hat{u}(\xi_1, ..., \xi_n)$ is the Fourier transform of $u(x_1, ..., x_n)$. We shall call x_i the space variables and ξ_i the dual variables.

The notion of pseudo-differential operators has grown in recent years out of attempt to obtain sharp *a priori* estimates for the solutions of the partial differential equations. Since its discovery it has been found to be one of the most powerful tools in attacking various problems in partial differential equations such as the existence and uniqueness of the boundary value problems [1], regularity of the solutions of the partial differential equations [2], solvability of a general partial differential operator [3], etc.

The basic calculus formulas for the pseudo-differential operators are due to J. Kohn and L. Nirenberg [4] as a development of the theory of Singular Integral Operators of Calderon and Zygmund. They constructed an algebra of pseudo-differential operators with symbols having bounded derivatives in the space variables and proved the boundedness of such operators. However their algebra is too restrictive for many purposes. One of the

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aims in considering pseudo-differential operators is to include inverses of a wide class of partial differential operators, but the only invertible operators in their algebra are the elliptic ones. Lars Hörmander [2] has considered more general classes of operators with symbols satisfying the following regular conditions:

$$|\partial_\xi^{\alpha}\partial_x^{\beta}p(x, \xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|} \quad \forall (x, \xi) \in K \times R^n, \quad 0 \leqslant \delta < \rho \leqslant 1$$

and proved the boundedness of such operators by using a tricky partition of unity. (A simpler proof of this can be found in his lecture on the Fourier Integral Operators.)

Concerning operators with nonregular symbols L. Nirenberg posed the following questions:

(1) Determine if the operator P can be extended as a bounded operator from $L^2(R^n)$ to $L^2(K)$ for any compact set $K \subset R^n$ if $p(x, \xi)$ belongs to $C^0(R^n \times R^n)$ and satisfies

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n} p(x, \xi)| \leqslant C_{\alpha K}(1 + |\xi|)^{-(\alpha_1 + \cdots + \alpha_n)} \; \forall x \in K, \xi \in R^n. \tag{1.1}$$

(2) Determine if the operator P can be extended as a bounded operator from $L^2(R^n)$ to $L^2(K)$ for any compact set $K \subset R^n$ if $p(x, \xi)$ belongs to $C^0(R^n \times R^n)$ and satisfies

$$\begin{aligned} &\text{a.}\quad p(x, t\xi) = p(x, \xi) \qquad \text{for all} \quad t > 0 \qquad \text{and} \qquad |\xi| \geqslant 1. \\ &\text{b.}\quad |\partial_{x_1}^{\alpha_1} \cdots \partial_{x_j}^{\alpha_j}\partial_{\xi_{j+1}}^{\beta_{j+1}} \cdots \partial_{\xi_n}^{\beta_n} p(x, \xi)| \leqslant C_{\alpha\beta K}(1 + |\xi|)^{-(\beta_{j+1}+\cdots+\beta_n)} \\ &\quad\;\; \text{for all} \quad x \in K \qquad \text{and} \qquad \xi \in R^n. \qquad \text{Here } j \text{ is fixed, } 0 < j < n. \end{aligned} \tag{1.2}$$

In this paper we shall answer these questions negatively. However, we can prove the boundedness of the operator if we strengthen the condition (1.1) slightly. We also point out that if j is equal to 0 or n in (1.2) the associated operators are bounded. This proposition will be proved in Section 3 and explains why we are interested in symbols with mixed smoothness.

The plan of this paper is as follows: In Section 2 we motivate the definition of the pseudo-differential operators and define some terms. Section 3 contains some proofs of the basic facts of the pseudo-differential operators. In Section 4 we present some unbounded operators with symbols satisfying (1.1) and prove that if some integrals of the symbols are uniformly bounded the associated operators are bounded. Unbounded operators with symbols satisfying (1.2) are given in Section 5.

I wish to thank Prof. L. Nirenberg and L. Hörmander for their suggestions and for their generous help.

2. Definitions and Notations

We shall write $x = (x_1, ..., x_n)$ for the coordinate in R^n and $\xi = (\xi_1, ..., \xi_n)$ for the dual coordinate. For an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n,$$

$$\partial_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \qquad \alpha! = \alpha_1! \cdots \alpha_n!$$

$$\partial_\xi^\alpha = \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{\alpha_n}, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

Definition 2.1. By $C_0^\infty(R^n)$ we denote the set of all infinitely differentiable functions with compact support in R^n.

Definition 2.2. By $C^\infty(R^n)$ we denote the set of all infinitely differentiable functions in R^n.

As the pseudo-differential operators are defined with the aid of Fourier transform, we recall the definition and some well-known facts:

$$\begin{aligned} \hat{u}(\xi) &= (2\pi)^{-n} \int e^{-ix\xi} u(x)\, dx \qquad \text{for} \quad u \in C_0^\infty(R^n), \\ \widehat{D^\alpha u}(\xi) &= \xi^\alpha \hat{u}(\xi), \\ u(x) &= \int e^{ix\xi} \hat{u}(\xi)\, d\xi. \end{aligned} \tag{2.1}$$

Definition 2.3. For any real s we denote by $H_s(R^n)$ the completion of $C_0^\infty(R^n)$ under the following norm:

$$\| u \|_s^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s\, d\xi.$$

The motivation of the definition of the pseudo-differential operators is easily seen if we represent the partial differential operators with the aid of Fourier transform. Let $p(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha$ and $u(x)$ belong to $C_0^\infty(R^n)$; the formulas (2.1) give

$$\begin{aligned} Pu(x) &= \sum_{|\alpha| \leqslant m} a_\alpha(x)\, D^\alpha \int e^{ix\xi} \hat{u}(\xi)\, d\xi \\ &= \sum_{|\alpha| \leqslant m} a_\alpha(x) \int e^{ix\xi} \xi^\alpha \hat{u}(\xi)\, d\xi \\ &= \int e^{ix\xi}\, p(x, \xi)\, \hat{u}(\xi)\, d\xi. \end{aligned}$$

To define pseudo-differential operators we use the same formula

$$Pu(x) = \int e^{ix\xi}\, p(x, \xi)\, \hat{u}(\xi)\, d\xi \tag{2.2}$$

with the functions $p(x, \xi)$ satisfying some special conditions on the growth of the partial derivatives.

Definition 2.4. By $S^m_{\rho\delta}(R^n)$ we denote the set of all functions $p(x, \xi)$ such that

$$| D_\xi^{\alpha} D_x^{\beta} p(x, \xi)| \leqslant C_{\alpha,\beta,K}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|} \qquad \forall x \in K, \quad \xi \in R^n,$$

where K is a compact subset of R^n.

Example 1. If $p(x, \xi) \in C^\infty(R^n \times R^n)$ and $p(x, \xi)$ is positively homogeneous of degree m, i.e., $p(x, t\xi) = t^m p(x, \xi)$, $t \geqslant 1$ and $|\xi| > 1$ we have $p(x, \xi) \in S^m_{10}(R^n)$.

Example 2. If $p(y) \in C_0^\infty(R^1)$, then $p(x, \xi) \in S^0_{11}(R^1)$.

Definition 2.5. A linear operator P from $C_0^\infty(R^n)$ to $C^0(R^n)$ is said to be of order m if it can be extended to a bounded operator from $H_m(R^n)$ to $H_0(K)$ where K is any compact subset of R^n.

3. Some Basic Facts about Pseudo-Differential Operators

Theorem 3.1. *If $p(x, \xi) \in S^m_{\rho\delta}(R^n)$ then* (2.2) *defines a linear operator P from $C_0^\infty(R^n)$ to $C^\infty(R^n)$.*

Proof. If $u \in C_0^\infty(R^n)$, then $|\hat{u}(\xi)|$ decreases faster than any power of $|\xi|$ as $|\xi|$ becomes large; thus for each integer N and for each u the estimate

$$| D_x^{\beta} p(x, \xi)| \leqslant C_{K\beta}(1 + |\xi|)^{m+\delta|\beta|-N}$$

holds for all (x, ξ). These estimates justify differentiation under the integral sign in (2.2). The resulting integrals are absolutely convergent, and the desired result follows immediately.

Similarly, we can prove the following theorem:

Theorem 3.2. *If $p(x, \xi)$ satisfies either* (1.1) *or* (1.2), (2.2) *defines a linear operator from $C_0^\infty(R^n)$ to $C^0(R^n)$.*

THEOREM 3.3. *If $p(x, \xi)$ has bounded partial derivatives with respect to the space variables in $K \times R^n$ for any compact subset K of R^n, the associated operator P is of order zero.*

This is proved in Theorem 1 of [4]. As a consequence, we have

COROLLARY 1. *If $p(x, \xi)$ satisfies (1.2) with $j = n$ the associated operator is of order zero.*

COROLLARY 2. *If $p(x, \xi)$ belongs to $S^0_{00}(R^n)$, P is of order zero.*

In fact, A. Calderón and R. Vaillancourt [5] proved the following stronger result:

Let the symbol $p(x, \xi)$ be a matrix of functions $p_{ij}(x, \xi)$ defined on $R^n \times R^n$ such that

$$| D_\xi^{\alpha} D_x^{\beta} p(x, \xi)| \leqslant C_{\alpha\beta}$$

for $\alpha_l, \beta_K = 0, 1, 2, 3$ and all x and ξ. Then the associated pseudo-differential operator can be extended to a bounded operator from $L^2(R^n)$ to $L^2(R^n)$.

Another elementary fact is

THEOREM 3.4. *If $p(x, \xi)$ belongs to $C^0(R^n \times R^n)$ and vanishes for large ξ, the associated operator P is of order zero.*

Proof. As $p(x, \xi)$ has compact support in ξ, we have

$$\int_K dx \int | p(x, \xi)|^2 \, d\xi \leqslant C.$$

From the fact that $| Pu(x)| \leqslant \int | p(x, \xi)| \, | \hat{u}(\xi)| \, d\xi$, we obtain

$$| Pu(x)|^2 \leqslant \int | p(x, \xi)|^2 \, d\xi \int | \hat{u}(\xi)| \, d\xi.$$

Hence

$$\int_K | Pu(x)|^2 \leqslant C \int | \hat{u}(\xi)|^2 \, d\xi$$

as required.

From Theorem 3.4 we know that the values of $p(x, \xi)$ for small ξ do not play any role in the boundedness of the associated operator P. For we can use a partition of unity to decompose the symbol $p(x, \xi)$ into a sum of $p_1(x, \xi)$ and $p_2(x, \xi)$ such that $p_1(x, \xi)$ has compact support in ξ and p_2 vanishes for small ξ.

By using the same method as in the proof of Theorem 3.4 we can prove that if $p(x, \xi)$ belongs to $C^\infty(R^n \times R^n)$ and has compact support in ξ then P is of order m for any m.

We also remark that the question 2 in Section 1 is equivalent to the following question for fixed j:

2′ Is the operator P of order zero if its symbol $p(x, \xi)$ satisfies

(a) $p(x, \xi)$ is continuous on the unit sphere and $p(x, t\xi) = p(x, \xi)$ for $t \geqslant 1$ and $|\xi| = 1$.

$$(3.2)$$

(b) $|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_j}^{\alpha_j} \partial_{\xi_{j+1}}^{\beta_{j+1}} \cdots \partial_{\xi_n}^{\beta_n} p(x, \xi)| \leqslant C_{\alpha\beta K}(1 + |\xi|)^{-(\beta_{j+1}+\cdots+\beta_n)}$ for all $x \in K$ and $|\xi| \geqslant 1$.

THEOREM 3.5. *If $p(x, \xi)$ satisfies* (1.2) *with $j = 0$, the associated operator is of order zero.*

We present a proof due to W. Littman.

Let $\gamma = |\xi|$ and $\xi = r\omega$; we can write

$$\begin{aligned}(Pu)(x) &= \int_{r\leqslant 1} e^{ix\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi + \int_{r\geqslant 1} e^{ix\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi \\ &= P_1 u + P_2 u\end{aligned}$$

It follows from the proof of Theorem 3.4 that

$$\int_K |P_1 u|^2\, dx \leqslant C \int |u|^2\, dx.$$

Let $G_k(\omega, \delta)$ be the Green's function of $\Delta^{(k)}$ on the unit sphere, where Δ is the Laplace Betrami operator on the unit sphere. G_k is continuous and bounded for large k. Now we are going to estimate $P_2 u$ as follows:

$$\begin{aligned}P_2 u(x) &= \int_1^\infty r^{n-1}\, dr \int e^{ix\xi} p(x, \omega)\, \hat{u}(\xi)\, d\omega \\ &= \int_1^\infty r^{n-1}\, dr \int e^{ix\xi} \hat{u}(\xi)\, d\omega \int_{|\delta|=1} G_k(\omega, \delta)(\Delta^{(k)} p(x, \delta))\, d\delta \\ &= \int_{|\delta|=1} \Delta^{(k)} p(x, \delta)\, d\delta \int_{|\xi|\geqslant 1} G_k(\xi, \delta)\, e^{ix\xi} \hat{u}(\xi)\, d\xi.\end{aligned}$$

Squaring both sides and applying Schwarz's inequality we obtain

$$|P_2 u(x)|^2 \leqslant C \int_{|\delta|=1} d\delta \left| \int_{|\xi|\geqslant 1} G_k(\xi, \delta)\, e^{ix\xi} \hat{u}(\xi)\, d\xi \right|^2$$

Integrating both sides gives

$$\int_K | P_2u(x)|^2 \, dx \leqslant C \int_{|\delta|=1} d\delta \int \left| \int_{|\xi|\geqslant 1} G_k(\xi, \delta) \, e^{ix\xi}\hat{u}(\xi) \, d\xi \right|^2 dx.$$

Applying Parseval's identity to the last integral we have

$$\int | P_2u(x)|^2 \, dx \leqslant C \int_{|\delta|=1} d\delta \int_{|\xi|\geqslant 1} | G_k(\delta, \xi)|^2 \, | \hat{u}(\xi)|^2 \, d\xi$$

$$\leqslant C \int | \hat{u}(\xi)|^2 \, d\xi.$$

The proof is complete.

4. On Inhomogeneous Nonregular Symbols

THEOREM 4.1. *Let $\chi(\xi)$ be a $C_0^\infty(R^n)$ function with support $\chi(\xi) \subset \{\xi \mid 1 \leqslant |\xi| \leqslant 5\}$ and χ is equal to one if $2 \leqslant |\xi| \leqslant 4$. Let η_k be a sequence in R^n such that $|\eta_k| = 3 \cdot 5^k$. Then the operator P associated with $p(x, \xi) = \sum_{k=1}^\infty a_k e^{i\eta_k x}\chi(5^{-k}\xi)$* [1] *is not bounded from $L^2(R^n)$ to $L^2(K)$ for any K with nonempty interior, if $\sum_1^\infty |a_k|^2$ is divergent.*

Proof. Assume the contrary that, for some constant C,

$$\| Pu \|_K \leqslant C \| u \|_{R^n} \tag{4.1}$$

Choose $\varphi \neq 0$ such that $\hat{\varphi} \in C_0^\infty(R^n)$ and supp $\hat{\varphi} \subset$ unit ball. Then we define

$$\hat{u}_m(\xi) = \sum_1^m b_k \hat{\varphi}(\xi - \eta_k) \tag{4.2}$$

As all terms in (4.2) have disjoint supports it follows that

$$\| \hat{u}_m(\xi) \|_{R^n} = \sum_1^m | b_k |^2$$

and

$$(Pu_m)(x) = \sum_1^m b_k a_k e^{-i\eta_k x} \int e^{ix\xi}\chi(5^{-k}\xi) \, \hat{\varphi}(\xi - \eta_k) \, d\xi$$

Since $\chi(5^{-k}\xi) = 1$ if $\hat{\varphi}(\xi - \eta_k) \neq 0$, we obtain

$$(Pu_m)(x) = \left(\sum_1^m b_k a_k \right) \varphi(x).$$

[1] Such expressions of $p(x, \xi)$ appeared first in [7].

Hence

$$\| Pu_m \|_k^2 = \left| \sum_1^m b_k a_k \right|^2 \int_K | \varphi(x)|^2 \, dx.$$

Applying (4.1) to u_m gives

$$\left| \sum_1^m b_k a_k \right|^2 \leqslant C \sum_1^m | b_k |^2.$$

We then have

$$\sum_1^\infty | a_k |^2 \leqslant C.$$

But this contradicts our hypothesis on a_k. The proof is complete.

COROLLARY 1. *There exist unbounded pseudo-differential operators from $L^2(R^n)$ to $L^2(K)$ with symbols satisfying* (1.1).

Proof. Define $p(x, \xi)$ as in Theorem 4.1 with $a_k = 1/\sqrt{k}$, i.e.,

$$p(x, \xi) = \sum_1^\infty \frac{1}{\sqrt{k}} e^{-i\eta_k x} \chi(5^{-k}\xi_1, \ldots, 5^{-k}\xi_n). \tag{4.3}$$

To prove p satisfies (1.1) we observe that the terms in (4.3) have disjoint supports and hence we have

$$\begin{aligned} | \partial_\xi{}^\alpha p(x, \xi)| &\leqslant \sup_k | 5^{-|\alpha| k} \chi(5^{-k}\xi)| \\ &\leqslant C(1 + | \xi |)^{|\alpha|} \sup_{\substack{\eta \\ |\beta| \leqslant \alpha}} | \eta^{(\beta)} \chi^{(\beta)}(\eta)| \\ &\leqslant C(1 + | \xi |)^{|\alpha|} \end{aligned}$$

as required. Thus the corollary follows from the theorem since $\sum_1^\infty | a_k |^2 = \infty$.

The symbols defined in this corollary belong to $S_{11}^0(R^n)$. The boundedness of operators with symbols in $S_{\rho,\delta}^0(R^n)$ for $0 \leqslant \delta < \rho \leqslant 1$ is proved in [2]. Kumano-go [6] constructed an unbounded pseudo-differential operator with symbol belonging to $\bigcap_{p<1} S_{p,1}^0$. Recently Hörmander [7] proves that if every operator with symbol $p(x, \xi) \in S_{\rho,\delta}^0$ is of order zero, then necessarily $\delta \leqslant \rho$. However, the question of the continuity of operators having symbols in $S_{\rho,\rho}^0$ with $0 < P < 1$ remains open.

The symbols defined in the theorem do not have bounded space derivatives at any point in R^n. So it is natural to ask whether the operator is bounded if its symbol $p(x, \xi)$ satisfies

$$| x_1^{\beta_1} \partial_x{}^\beta \partial_\xi{}^\alpha p(x, \xi)| \leqslant C_{\alpha\beta K}(1 + | \xi |)^{-|\alpha|}, \qquad \forall (x, \xi) \in K \times R^n.$$

The answer to this question which was raised by Hörmander is still not known.

We can show that a slight strengthening of condition (1.1) yields boundedness:

THEOREM 4.2. *If*

$$\int \frac{|p(x,\xi)|^2}{1+|\xi|}\,d\xi \qquad \textit{and} \qquad \int (1+|\xi|)\,|p_\xi(x,\xi)|^2\,d\xi$$

are bounded functions of $x \in R^1$ *then* P *is of order zero.*

Proof. Without loss of generality we can assume $p(x, 0) = 0$. Let Q be the adjoint operator of P. We have

$$\widehat{Qf}(\xi) = \int_K e^{-ix\xi}\overline{p(x,\xi)}\,f(x)\,dx, \qquad f \in C_0^\infty(K).$$

Squaring both sides gives

$$\begin{aligned}|\widehat{Qf}(\xi)|^2 &= \int_K \int_K e^{i(x-y)\xi} p(x,\xi)\,\overline{p(y,\xi)}\,\overline{f(x)}\,f(y)\,dx\,dy \\ &= \int_K \overline{f(x)}\,dx \int_K e^{i(x-y)\xi} f(y)\,dy \int_0^\xi (p(x,\eta)\,\overline{p_\eta(y,\eta)} \\ &\qquad + p_\eta(x,\eta)\,p(y,\eta))\,d\eta\end{aligned}$$

and by the inequality $2\,|ab| \leqslant |a|^2 + |b|^2$

$$\begin{aligned}|\widehat{Qf}(\xi)|^2 \leqslant \int \frac{d\eta}{1+|\eta|}\left|\int_K e^{ix\xi}p(x,\eta)\,f(x)\,dx\right|^2 \\ + \int (1+|\eta|)\,d\eta \left|\int_K e^{-iy\xi}p_\eta(y,\eta)\,f(y)\,dy\right|^2.\end{aligned}$$

Integrating both sides and then using Parseval's identity, we obtain

$$\begin{aligned}\int |\widehat{Qf}(\xi)|^2\,d\xi &\leqslant \int \frac{d\eta}{1+|\eta|}\int\left|\int_K e^{ix\xi}p(x,\eta)\,f(x)\,dx\right|^2 d\xi \\ &\quad + \int (1+|\eta|)\,d\eta \int\left|\int_K e^{-iy\xi}p_\eta(y,\eta)\,f(y)\,dy\right|^2 d\xi, \\ \int |\widehat{Qf}|^2\,d\xi &\leqslant \int \frac{d\eta}{1+|\eta|}\int_K |p(x,\eta)|^2\,|f(x)|^2\,dx \\ &\quad + \int (1+|\eta|)\,d\eta \int_K |p(x,\eta)\,f(x)|^2\,dx \\ &\leqslant C\int |f(x)|^2\,dx.\end{aligned}$$

Since P and Q have the same operator bound. The proof is complete.

Application: If $f(y)$ belongs to $C_0^\infty(R^1)$, the above theorem shows that the operator with the symbol $f(x\xi)$ is of order zero.

Theorem 4.2 can be extended to R^n as follows:

THEOREM 4.3. *Denote* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ *and* $\beta = (\beta_{j+1}, ..., \beta_n)$ *be subsets of* $\{1, 2, ..., n\}$ *such that* $\alpha \cap \beta = \phi$. *Then the operator with symbol* $p(x, \xi)$ *is bounded if*

$$\int \frac{1 + |\xi_{\beta_{j+1}} \cdots \xi_{\beta_n}|}{1 + |\xi_{\alpha_1} \cdots \xi_{\alpha_k}|} \left| \frac{\partial^{n-j} p(x, \xi)}{\partial \xi_{\beta_{j+1}} \cdots \partial \xi_{\beta_n}} \right|^2 d\xi \leqslant C, \qquad \forall x \in K.$$

Proof. Again we can assume $p(x, \xi) = 0$ if $\xi_i = 0$ for some i. Let Q be the adjoint operator of P. We have

$$\begin{aligned} Qf(\xi) &= \int_K \int_K e^{i(x-y)\xi} \int_0^\xi \frac{\partial^n p(x, \eta)\, \overline{p(y, \eta)}}{\partial \eta_1 \cdots \partial \eta_n}\, d\eta f(x) f(y)\, dx\, dy \\ &= \sum_{\alpha,\beta} \int_K f(x)\, dx \int_K f(y)\, e^{i(x-y)\xi}\, dy \int_0^\xi \frac{\partial^k p(x, \eta)}{\partial \eta_\alpha} \frac{\partial^{n-k} \overline{p(y, \eta)}}{\partial \eta_\beta}\, d\eta. \end{aligned}$$

In virtue of the assumptions on $p(x, \xi)$ we can proceed as in the proof of the Theorem 4.2 to conclude the boundedness of the operator P. The proof is complete.

Remark. The result of Theorem 4.2 is rather sharp if we compare it with Theorem 4.1 with $a_k = 1/\sqrt{k}$. Since the functions P and ξP_ξ behave like $1/\sqrt{k}$ when $\xi \sim 5^k$ if $a_k = 1/\sqrt{k}$, consequently $p(x, \xi) \sim 1/\sqrt{\log \xi}$. But Theorem 4.2 tells us that if $p(x, \xi)$ and $\xi p_\xi(x, \xi)$ behave like $(\log \xi)^{-1/2-\delta}$ for $\delta > 0$ the operator P is of order zero.

5. On Homogeneous Nonregular Symbols

We now present an example showing that the answer to question (2) is in the negative:

THEOREM 5.1. *Define* $q(x_1, x_2, \xi_1, \xi_2) = p(x_2, \xi_2/\xi_1)$ *where* $p(x, \xi)$ *is the function defined in Theorem* 4.1 *for the one-dimensional case. Let L be a compact subset of* R^1 *and* $K = [1, 1 + \frac{1}{4}] \times L$. *Then the operator* Q *is not bounded from* $L^2(R^n)$ *to* $L^2(K)$ *if* $\sum |a_k|^2$ *is divergent.*

Proof. Assume the contrary there exists a constant C such that

$$\| Qu \|_K \leqslant C \| u \|_{R^2}. \tag{5.1}$$

We now define

$$\hat{u}_m(\xi_2) = \sum_1^m b_k \hat{\varphi}(\xi_2 - 3 \cdot 5^k),$$

$$\hat{w}_m(\xi_1, \xi_2) = \hat{h}(\xi_1)\, \hat{u}_m(\xi_2),$$

where $\hat{h} \in C_0^\infty[1, 1 + \epsilon]$ and $\hat{\varphi} \in C_0^\infty[0, 1]$. As the support of $\hat{w}_m(\xi_1, \xi_2)$ is a subset of $[1, 1 + \frac{1}{4}] \times [3 \cdot 5^k, 3 \cdot 5^k + 1]$, $\hat{w}_m(\xi_1, \xi_2)$ is not zero only if

$$\xi_2/\xi_1 \in [2 \cdot 5^k, 3 \cdot 5^k + 1].$$

According to the definition of χ, supp $\hat{h}(\xi_1)\, \hat{\varphi}(\xi_2 - 3 \cdot 5^k)$ and supp $\chi(5^{-j}\xi_2/\xi_1)$ are disjoint if $j \neq k$. Thus we have

$$Qw_m = \sum_1^m b_k a_k e^{-3 \cdot 5^k x_2} \int e^{ix_1\xi_1} \hat{h}(\xi_1)\, d\xi_1 \int e^{ix_2\xi_2} \chi(5^{-k}\xi_2/\xi_1)\, \hat{\varphi}(\xi_2 - 3 \cdot 5^k)\, d\xi_2 .$$

But if $(\xi_1, \xi_2) \in \operatorname{supp} \hat{h}(\xi_1)\, \hat{\varphi}(\xi_2 - 3 \cdot 5^k)$, $1 = \chi(5^{-k}\xi_2/\xi_1)$. We obtain

$$Qw_m = \sum_1^m a_k b_k h(x_1)\, \varphi(x_2)$$

or

$$\| Qw_m \|^2 = \left(\sum_1^m a_k b_k \right)^2 \int |\, h\,|^2\, |\, \varphi\,|^2\, dx_1\, dx_2 .$$

In view of (5.1) we conclude that

$$\sum a_k^2 \leqslant C.$$

This contradicts our assumption on a_k. Thus the assumption in the beginning of the proof is absurd and the proof is complete.

COROLLARY. *There exists an unbounded operator Q from $L^2(R^n)$ to $L^2(K)$ with symbol satisfying* (3.2).

Proof. Define Q as in the Theorem 5.1 with $a_k = 1/\sqrt{k}$. It is obvious that $q(x, \xi)$ is homogeneous of degree zero. The continuity of $q(x, \xi)$ on the unit sphere follows from the fact that $q(x, \xi)$ behaves like $1/\sqrt{\log \xi_2}$ as (ξ_1, ξ_2) tends to $(1, 0)$. Noticing that

$$\begin{aligned}\partial_{\xi_1}^{\alpha} q(x, \xi_1) &= \sum_1^{\infty} a_k e^{-i\eta_k x_\alpha} \frac{5^{-k\alpha}}{\xi_2{}^{\alpha}} \chi^{(\alpha)} \left(5^{-k} \frac{\xi_1}{\xi_2}\right) \\ &= \frac{1}{\xi_1{}^{\alpha}} \sum_1^{\infty} a_k e^{-i\eta_k x_2} \frac{(5^{-k}\xi_1)^{\alpha}}{\xi_2{}^{\alpha}} \chi^{(\alpha)} \left(5^{-k} \frac{\xi_1}{\xi_2}\right)\end{aligned}$$

we then have $\xi_1{}^{\alpha} \partial_{\xi_1}^{\alpha} q \sim 1/\sqrt{\log \xi_2}$ as (ξ_1, ξ_2) tends to $(1, 0)$. Hence $| \xi |^{\alpha} \partial_{\xi_1}^{\alpha} q(x_1, x_2, \xi_1, \xi_2)$ are bounded. The proof is complete.

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